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1992 J. Phys. A: Math. Gen. 25 L1055

(http://iopscience.iop.org/0305-4470/25/17/007)

J. Phys. A: Math. Gen. 25 (1992) L1055–L1058. Printed in the UK

LETTER TO THE EDITOR

Reciprocal logarithmic time-dependence for a simple one-dimensional random walk

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Received 22 April 1992

Abstract. It is shown for a simple one-dimensional random walk that the probability of the existence of a point visited exactly once, with both its neighbours visited at least once, has a time-dependence of $\sim 2/\ln t$ as $t \to \infty$.

In this letter I present the following property of a simple one-dimensional random walk with independent increments [1, 2]

$$f(t) = P\left(\min_{L_t < m < R_t} N(t, m) = 1\right) \sim \frac{2}{\ln t} \qquad t \to \infty \tag{1}$$

where $P(\ldots)$ denotes 'the probability of ...', L_t and R_t are respectively the left and right boundaries of the set of visited points

$$L_t = \min_{0 \leqslant \tau \leqslant t} S_\tau \qquad R_t = \max_{0 \leqslant \tau \leqslant t} S_\tau \tag{2}$$

$N(t, m)$ is the number of visits to a point m

$$N(t, m) = \|\{\tau : 0 \leqslant \tau \leqslant t, S_\tau = m\}\| \tag{3}$$

and a sequence of random quantities S_t is such that

$$S_0 = 0 \qquad S_t = S_{t-1} + X_t \qquad t = 1, 2, \ldots \tag{4}$$

where independent random quantities X_t have the distribution

$$P(X_t = 1) = P(X_t = -1) = \tfrac{1}{2} \qquad t = 1, 2, \ldots. \tag{5}$$

To prove (1) we consider the functions of integer $t, m, n \geqslant 0$:

$$f_2(t; m, n) = P(m = R_t - S_t, n = S_t - M_t, D_t \neq \varnothing) \qquad m \geqslant 0, n \geqslant 1, t \geqslant 0 \tag{6a}$$

$$f_2(t; m, 0) = 0 \qquad m \geqslant 0, t \geqslant 0 \tag{6b}$$

$$f_1(t; m) = P(m = R_t - S_t, D_t = \varnothing) \qquad m \geqslant 1, t \geqslant 0 \tag{7a}$$

$$f_1(t; 0) = P(R_t = S_t, D_t = \varnothing, \nu(t, R_t) > 1) \qquad t \geqslant 1, f_1(0; 0) = 1 \tag{7b}$$

$$f_0(t) = P(R_t = S_t, D_t = \varnothing, \nu(t, R_t) = 1) \qquad t \geqslant 1, f_0(0) = 0 \tag{8}$$

where

$$D_t = \{m : 0 < m < R_t, \nu(t, m) = 1\} \tag{9}$$

and in the case of $D_t \neq \varnothing$

$$M_t = \inf(D_t). \tag{10}$$

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Considering one step and making use of the distribution (5), we have the following equations:

$$2f_2(t;\, m, n) = f_2(t-1;\, m+1, n-1) + f_2(t-1;\, m-1, n+1) \qquad m \geq 1, n \geq 1, t \geq 1 \tag{11a}$$

$$f_2(t;\, m, 0) = 0 \qquad m \geq 0, t \geq 0 \tag{11b}$$

$$2f_2(t;\, 0, n) = f_2(t-1;\, 0, n-1) + f_2(t-1;\, 1, n-1) \qquad n \geq 2, t \geq 1 \tag{11c}$$

$$2f_2(t;\, 0, 1) = f_0(t-1) \qquad t \geq 1 \tag{11d}$$

$$f_2(0;\, m, n) = 0 \qquad m \geq 0, n \geq 0 \tag{11e}$$

$$2f_1(t;\, m) = f_1(t-1;\, m+1) + f_1(t-1;\, m-1) + f_2(t-1;\, m-1, 1) \qquad m \geq 2, t \geq 1 \tag{12a}$$

$$2f_1(t;\, 1) = f_1(t-1;\, 2) + f_1(t-1;\, 0) + f_2(t-1;\, 0, 1) + f_0(t-1) \qquad t \geq 1 \tag{12b}$$

$$2f_1(t;\, 0) = f_1(t-1;\, 1) \qquad t \geq 1 \tag{12c}$$

$$f_1(0;\, m) = 0 \qquad m \geq 1 \tag{12d}$$

$$f_1(0;\, 0) = 1 \tag{12e}$$

$$2f_0(t) = f_1(t-1;\, 0) \qquad t \geq 1 \tag{13a}$$

$$f_0(0) = 0. \tag{13b}$$

By definition (6), we find

$$P(D_t \neq \varnothing) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_2(t;\, m, n) \qquad t \geq 0. \tag{14}$$

From the symmetry of a simple random walk it follows that

$$P(D'_t \neq \varnothing) = P(D_t \neq \varnothing) \tag{15}$$

where

$$D'_t = \{m\colon L_t < m < 0,\ \nu(t, m) = 1\}. \tag{16}$$

It is obvious that either $D_t = \varnothing$ or $D'_t = \varnothing$, hence

$$f(t) = P(D_t \neq \varnothing \text{ or } D'_t \neq \varnothing) = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_2(t;\, m, n) \qquad t \geq 0. \tag{17}$$

The system of equations (11), (12), (13), (17) allows us to calculate $f(t)$ respectively. Now we consider the generating functions:

$$F_2(z;\, m, n) = \sum_{t=0}^{\infty} z^t f_2(t;\, m, n) \tag{18a}$$

$$F_1(z;\, m) = \sum_{t=0}^{\infty} z^t f_1(t;\, m) \tag{18b}$$

$$F_0(z) = \sum_{t=0}^{\infty} z^t f_0(t) \tag{18c}$$

$$F(z) = \sum_{t=0}^{\infty} z^t f(t) \tag{18d}$$

where $|z| < 1$.

Summing equations (11), (12), (13), (17) with z', we obtain

$$(2/z)F_2(z;\,m,n)=F_2(z;\,m+1,n-1)+F_2(z;\,m-1,n+1) \qquad m\geqslant 1, n\geqslant 1 \tag{19a}$$

$$F_2(z;\,m,0)=0 \qquad m\geqslant 0 \tag{19b}$$

$$(2/z)F_2(z;\,0,n)=F_2(z;\,0,n-1)+F_2(z;\,1,n-1) \qquad n\geqslant 2 \tag{19c}$$

$$(2/z)F_2(z;\,0,1)=F_0(z) \tag{19d}$$

$$(2/z)F_1(z;\,m)=F_1(z;\,m+1)+F_1(z;\,m-1)+F_2(z;\,m-1,1) \qquad m\geqslant 2 \tag{20a}$$

$$(2/z)F_1(z;\,1)=F_1(z;\,2)+F_1(z;\,0)+F_2(z;\,0,1)+F_0(z) \tag{20b}$$

$$(2/z)F_1(z;\,0)=F_1(z;\,1)+2/z \tag{20c}$$

$$\lim_{m\to\infty} F_1(z;\,m)=0 \tag{20d}$$

$$(2/z)F_0(z)=F_1(z;\,0) \tag{21}$$

$$F(z)=2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_2(z;\,m,n). \tag{22}$$

Condition (20*d*) which is necessary to choose the only solution follows from the simple property $f_1(t;\,m)=0$ at $t<m$.

The system of equations (19)–(21) can be solved by standard methods of the finite difference calculus [3]. We show the main points of solution. First, we write the general solution of equation (19*a*) in the form

$$\tilde{F}_2(z;\,m,n)=C_1(z;\,m+n)x^n+C_2(z;\,m+n)x^{-n} \tag{23}$$

where $C_1(z;\,m)$, $C_2(z;\,m)$ are arbitrary functions and

$$x=\frac{1}{z}(1+\sqrt{1-z^2}). \tag{24}$$

Then condition (19*b*) gives

$$C_1(z;\,m)=-C_2(z;\,m) \qquad m\geqslant 0. \tag{25}$$

Further, equation (19*c*) transforms into

$$C_1(z;\,n)=C_1(z;\,n-1)\frac{u(x;\,n-1)}{u(x;\,n+1)} \qquad n\geqslant 2 \tag{26}$$

where

$$u(x;\,n)=\sinh[n\ln(x)] \tag{27}$$

and condition (19*d*) lets us find

$$C_1(z;\,1)=\frac{F_0(z)}{2u(x;\,2)}. \tag{28}$$

Thus, we can express the function $f_2(z;\,m,n)$ in terms of $F_0(z)$:

$$F_2(z;\,m,n)=\frac{u(x;\,1)u(x;\,n)}{u(x;\,m+n+1)u(x;\,m+n)}F_0(z). \tag{29}$$

To find the general solution of non-homogeneous equation (20a) we use the Green function

$$G(z;k;m)=-\frac{|u(x;m-k)|}{2u(x;1)} \tag{30}$$

which is a solution of the equation

$$(2/z)G(z;k;m)=G(z;k;m+1)+G(z;k;m-1)+\delta_{m,k} \tag{31}$$

and adding the general solution of homogeneous equation, we have

$$\tilde{F}_1(z;m)=\sum_{k=2}^{\infty}G(z;k;m)F_2(z;k-1,1)+C_3(z)x^m+C_4(z)x^{-m} \tag{32}$$

where $m\geqslant 1$ and $C_3(z)$, $C_4(z)$ are arbitrary functions.

Finally, making use of (20b)-(20d), (21) one can find that

$$F_1(z;m)=\frac{2}{z}\frac{x^{-m}}{x-1}-F_0(z)\sum_{k=m}^{\infty}\frac{u(x;k-m)u(x;1)}{u(x;k+1)u(x;k)} \qquad m\geqslant 1 \tag{33a}$$

$$F_1(z;0)=\frac{2}{z}F_0(z) \tag{33b}$$

$$F_0(z)=\frac{2}{x-1}\left(2+z\sum_{k=1}^{\infty}\frac{u(x;1)}{u(x;k)}\right)^{-1}. \tag{33c}$$

Taking the sum in equation (22), we obtain

$$F(z)=\frac{1}{x-1}F_0(z)=\frac{2}{(x-1)^2}\left(2+z\sum_{k=1}^{\infty}\frac{u(x;1)}{u(x;k)}\right)^{-1}. \tag{34}$$

Now we consider asymptotics at small positive $\varepsilon=1-z$, $\varepsilon\to 0$:

$$-1+x\sim\ln(x)\sim\sqrt{2\varepsilon} \tag{35}$$

$$\sum_{k=1}^{\infty}\frac{u(x;1)}{u(x;k)}\sim\int_1^{\infty}\frac{\sinh(\sqrt{2\varepsilon})}{\sinh(k\sqrt{2\varepsilon})}\,\mathrm{d}k=-\frac{\sinh(\sqrt{2\varepsilon})}{\sqrt{2\varepsilon}}\ln\left[\tanh\left(\frac{\sqrt{2\varepsilon}}{2}\right)\right]\sim\tfrac{1}{2}\ln\varepsilon \tag{36}$$

$$F(1-\varepsilon)\sim-\frac{2}{\varepsilon\ln\varepsilon}. \tag{37}$$

Taking the asymptotic of $f(t)$ at $t\to\infty$ in the form

$$f(t)\sim\frac{\varphi}{t^{\alpha}(\ln t)^{\beta}} \tag{38}$$

we have at $\alpha<1$

$$F(1-\varepsilon)\sim\varphi\sum_{t=2}^{\infty}\frac{(1-\varepsilon)^t}{t^{\alpha}(\ln t)^{\beta}}\sim\varphi\int_1^{\infty}\frac{\mathrm{e}^{-\varepsilon t}\,\mathrm{d}t}{t^{\alpha}(\ln t)^{\beta}}\sim\frac{\varphi\Gamma(1-\alpha)}{\varepsilon^{1-\alpha}(-\ln\varepsilon)^{\beta}}. \tag{39}$$

At last, comparing two asymptotics (37) and (39), we find $\varphi=2$, $\alpha=0$, $\beta=1$.

References

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